International Journal of Theoretical Physics, Vol. 19, No. 7, 1980

Fundamental Principles of Quantum Theory

Slawomir Bugajski

Institute of Physics, Silesian University, 40-007 Katowice, Poland

Pekka J. Lahti[1]

Department of Physical Sciences, University of Turku, 20500 Turku 50, Finland

Received December 12, 1979

After introducing general versions of three fundamental quantum postulates—the superposition principle, the uncertainty principle, and the complementarity principle—we discuss the question of whether the three principles are sufficiently strong to restrict the general Mackey description of quantum systems to the standard Hilbert-space quantum theory. We construct an example which shows that the answer must be negative. We introduce also an abstract version of the projection postulate and demonstrate that it could serve as the missing physical link between the general Mackey description and the standard quantum theory.

1. INTRODUCTION

The superposition principle, the uncertainty principle, and the complementarity principle form a very essential part of the traditional quantum theory. In spite of their foundational status in the theory, only the superposition principle has been appreciated in the quantum logic approach to axiomatic quantum mechanics, whereas the other two principles have been, to a large extent, neglected in this approach.

In this paper we are concerned with the question of whether the standard Hilbert space quantum theory can be erected on these fundamental principles only. More specifically, we shall study the question whether these principles (suitably formulated) are enough to restrict the general

[1]During the course of this work the author was invited to visit Silesian University in Katowice. The hospitality of the hosts is warmly acknowledged. The financial support of the University of Turku Foundation is also gratefully acknowledged.

0020-7748/80/0700-0499$03.00/0

Mackey axiomatics to the standard Hilbert-space quantum theory. With a counterexample this question will be answered in the negative. This negative result leads us to two alternative definitions of the proper quantum mechanical description which are the general Mackey axiomatics supplemented with the three principles and the standard Hilbert-space quantum theory.

Accepting the first of the above two definitions we think that our counterexample is a decisive indication that Hilbert space is too rich a structure to reflect properly the very ideas of the theory. Really, it is a well-known fact that many of the results of the Hilbert-space operator theory which are relevant for quantum mechanics can be reached in a much more general framework. We recall only the celebrated von Neumann–Varadarajan theorem (1968) and the spectral theory of Catlin (1968). With this definition, however, we face the problem of developing a new representation theory for the resulting abstract quantum theory. We do not touch this problem here.

Accepting the second definition for the proper quantum mechanical description, we advocate the view that the above three principles do not exhaust the foundations of the theory, but that there exists at least one more so-called fundamental principle which is an essential building stone of the theory. We are thus led to look for that (those) missing physical link(s). An analysis of the relevant literature has revealed that the von Neumann projection postulate (in one or another of its disguised forms) is one of the most important constituents in any representation theory which is intended to solve the problem of deriving the famous Hilbert-space axiom of Mackey (Bugajska and Bugajski, 1973b). Thus we guess that the projection postulate is a good candidate for such a missing physical link. It really turns out that the projection postulate (with a suitable formulation) is almost all we need to reach the standard Hilbert-space quantum theory from the general Mackey axiomatics. Only to avoid a trivial case, corresponding to the classical theory, something else is needed. This gap can be filled, e.g., with any of the three principles mentioned. In this approach, however, the above three fundamental principles have a very modest position, namely, to exclude only a trivial case. This is unsatisfactory because of the (traditionally) foundational status of the three principles in general discussion on the foundations of quantum theory (see, e.g., Jammer, 1966, 1974).

The structure of the paper is the following. We begin with a short sketch of the Mackey axiomatics, fixing thus our general set of frames—called the Mackey description. To appreciate our general problem we then proceed by formulating the three principles in the Mackey description. This is done in sections 3, 4, and 5. In section 6 we then ascertain that our formulations of the principles really are of quantal nature, i.e., that each of them excludes the classical description. Moreover, their equal foundational status is guaranteed by the logical independence of their formulations. In section 7 we finally are in a position to give our counterexample with which the negative answer to our main question is provided. Thereafter the two alternative definitions of the proper quantum mechanical description are formulated. In section 8 we formulate the projection postulate in our framework, and show that the Mackey description supplemented with the projection postulate is, apart from a trivial case, already the standard Hilbert-space quantum theory. We close this paper with a discussion of the results reached.

2. MACKEY DESCRIPTION

To keep our considerations as general as possible we start with the Mackey assumptions (1963), which seem rather unquestioned and applicable to any probabilistic physical theory.

Thus we assume that a theoretical description of a physical system is based on two sets: **O**—the set of all physical quantities (observables) concerning the system, and **S**—the set of all states (preparation procedures) of the system. We assume further that there is defined a function $p:\mathbf{O}\times\mathbf{S}\times\mathbf{B}(\mathbb{R})\to[0,1]$, where $\mathbf{B}(\mathbb{R})$ is the family of all Borel subsets of the real line $\mathbb{R}$, and $[0,1]$ is the unit interval. The number $p(A,\alpha,X)$ with $A\in\mathbf{O}$, $\alpha\in\mathbf{S}$, $X\in\mathbf{B}(\mathbb{R})$ is interpreted as the probability that a measurement of the quantity A on the system prepared in the state α will yield a result in X. This interpretation of the function p requires that for each fixed A in **O** and for each fixed α in **S** the set function $p(A,\alpha,\cdot):\mathbf{B}(R)\to[0,1]$ is a probability measure on $\mathbf{B}(\mathbb{R})$.

The Mackey assumptions (Mackey, 1963; see also Maczynski, 1967) concerning the properties of $(\mathbf{O},\mathbf{S},p)$ lead to a more fundamental set **L** (the logic of the considered system) of elementary observables (questions) which carries a natural structure of an orthomodular σ-complete poset. Each element A of **O** can be described as an **L**-valued measure on $(\mathbb{R},\mathbf{B}(\mathbb{R}))$, and each state α of **S** as a probability measure on **L**. The spectrum of A will be denoted by $\sigma(A)$. The family of **L**-valued measures corresponding to all observables of **O** is surjective, and the family of probability measures on **L** corresponding to all states of **S** is order-determining (full). The original function p can now be expressed as $p(A,\alpha,X) =\alpha\circ A(X)$, where $A\in\mathbf{O}$ is understood as an **L**-valued measure on $(\mathbb{R},\mathbf{B}(\mathbb{R}))$, and $\alpha\in\mathbf{S}$ as a probability measure on **L**. Thus we see that $(\mathbf{O},\mathbf{S},p)$ can be reconstructed from a more fundamental description $(\mathbf{L},\mathbf{S})$, where **S** is a full set of probability measures on **L**.

For the purpose of further considerations we have to assume that there is a sufficiently numerous set **P** of pure states in **S**. Such an assumption does not affect essentially the generality of our considerations. This is because the set **S** may be assumed to be a convex compact subset (in the pointwise convergence topology, see, e.g., Fischer and Rüttimann, 1978) of the family of real functions on **L**, in which case it possesses extremal points (see, e.g., Robertson and Robertson, 1966, p. 138). These extremal points can be considered as some idealized (pure) states on **L**, moreover for any $a \in \mathbf{L}$, $a \neq 0$, there exists a pure state, say α, such that $\alpha(a) = 1$. Thus the set **P** of pure states on **L** is unital, and hence also full (Pool, 1968).

The pair (**L**, **S**), fundamental for our discussion, with the above-stated properties, will be called the Mackey description.

3. THE SUPERPOSITION PRINCIPLE

The superposition principle in quantum theory can be formally considered in two different ways: either as a requirement of linearity of the quantum equations of motion, or as an assumption expressing the linearity of the underlying Hilbert space. The first aspect is rather questionable because of the growing tendency to introduce nonlinear terms into the Schrödinger equation (see, e.g., Mielnik, 1974; Haag and Bannier, 1978). The nonlinear generalizations of quantum mechanics operate on the same linear space of Ψ functions as the standard quantum mechanics; nevertheless, the second meaning of the superposition principle becomes less fundamental from the physical point of view. Anyway, the linearity of the space of Ψ functions is still considered traditionally as one of the most basic properties of quantum theory. This point of view, based on the matter-wave hypothesis of Louis de Broglie in 1924 and the wave mechanics of Erwin Schrödinger in 1926, is fully exposed in Paul Dirac's "Principles" in 1930, where one can find statements like: "each state of a dynamical system at a particular time corresponds to a ket vector, the correspondence being such that if a state results from the superposition of certain other states, its corresponding ket vector is expressible linearly in terms of the corresponding ket vectors of the other states, and conversely" (Dirac, 1958, p. 16).

Thus we have to define the notion of superposition, and to formulate the superposition principle in the general frame of the Mackey description. Observe that in this framework, where the standard one-to-one correspondence between pure states (one-dimensional subspaces of the Hilbert space) and atoms of **L** (projections with one-dimensional ranges) is not assumed, one can consider independently superpositions of pure states in **P** and superpositions of atoms in **L**.

The following definition of the notion of superposition of pure states, due to Varadarajan (1968, p. 116), is the most popular one:

(S) A pure state α is a superposition of pure states α_1 and α_2 iff $\alpha_1(a)=\alpha_2(a)=0$ implies $\alpha(a)=0$ for every $a\in\mathbf{L}$.

The Varadarajan notion of superposition is restated and considered by many authors; see, e.g., Bugajska and Bugajski (1973b), Gudder (1970), Zabey (1975). The definition of superposition recently given by Cantoni (1976) can easily be shown to be exactly equivalent to (S). A more specific definition has been proposed by Deliyannis (1976), who introduces a numerical characterization of a proportion the two pure states are superposed in. To do this he must introduce a new element into the Mackey description representing the "coupling" of pure states. Its physical meaning is rather hard to explain. On the other hand, axiom (ix)-b of Deliyannis shows explicitly that any superposition in his sense is also a superposition in the sense of (S). Thus (S) is rather sufficiently general and it expresses the very essence of the traditional notion of quantum superposition.

Observe that if the mentioned correspondence between pure states and atoms holds then (S) can be formulated as follows:

(S′) An atom a is a superposition of atoms a_1 and a_2 iff $a\leqslant b$ for every $b\in\mathbf{L}$ such that $b\geqslant a_1$ and $b\geqslant a_2$.

If $\mathbf{L}$ is lattice the condition (S′) can be formulated in terms of the lattice join: a is a superposition of a_1 and a_2 iff $a\leqslant a_1\vee a_2$. In the general case where the correspondence between atoms and pure states does not necessarily hold (S) and (S′) are independent and provide two alternative formulations of the notion of superposition. Something like (S′) is suggested by Jauch (Jauch, 1968, p. 106), but he prefers (without a sufficient justification) a stronger definition: an atom a is a (quantum) superposition of atoms a_1 and a_2 iff $a_1\vee a_2=a_1\vee a=a_2\vee a$. Note that any of the mentioned definitions provides an abstract counterpart of the linearity requirement of Dirac.

Now we can state the superposition principle requiring simply that for any two pure states (or for any two atoms) there is a third one which is their superposition. Thus we have the following:

(SP) For any two pure states there exists a third one which is a superposition of them according to (S).

(SP′) For any two atoms there exists a third one which is a superposition of them according to (S′).

Of course, the superposition principle can be valid only for irreducible systems, as superselection rules impose serious limitations on the validity of any form of the superposition principle. Note that Varadarajan (1968; see also Gudder, 1970; Zabey, 1975) formulates a stronger form of the superposition principle based on (S) demanding the lattice of all (S)-closed subsets of **P** to be exactly isomorphic to **L**. He ascribes to Dirac this point of view. Our (SP) was formulated by Pulmannova (1976).

4. THE UNCERTAINTY PRINCIPLE

The uncertainty principle originated through the work of Werner Heisenberg (1927). His point of departure was a reinterpretation of classical concepts, like position, velocity, and energy, in quantum domain by reducing the definability of a physical concept to its measurability. Heisenberg ended then with the conclusion that "all the concepts that are used in the classical theory for the description of a mechanical system can also be defined exactly for atomic processes." "But," he continues, "the experiments which allow such definitions carry with them an uncertainty if they involve the simultaneous determination of two canonically conjugate quantities" (Heisenberg, 1927, p. 179). Thus according to Heisenberg a consistent application of the concepts of classical physics in the quantum domain was secured by posing some limitations on the simultaneous measurability of certain physical quantities. These limitations Heisenberg expressed in his famous uncertainty relations.

In spite of some ambiguity in Heisenberg's writings, and the existence of at least two different ways of understanding his uncertainty principle, it is commonly accepted that the characteristic feature of quantum mechanics responsible for the uncertainty principle can be formulated in the following way (Lahti, 1979):

(UP) There exists at least one pair of observables A, B in **O** and a positive number h, such that for any state $\alpha \in \mathbf{S}$, for which the variances of A and B are well defined, the inequality $\mathrm{Var}(A,\alpha)\cdot\mathrm{Var}(B,\alpha) \geqslant h$ holds.

We will refer to (UP) as to the uncertainty principle. It expresses the intuitive idea of Heisenberg as formulated, e.g., in his Chicago Lectures: "...in many cases it is impossible to obtain an exact determination of the simultaneous values of two variables, but rather that there is a lower limit to the accuracy with which they can be known...this lower limit to the accuracy with which certain variables can be known simultaneously may be postulated as a law of nature..." (Heisenberg, 1949, p. 3).

5. THE COMPLEMENTARITY PRINCIPLE

The origin of Bohr's notion of complementarity was in his final acceptance of the wave–particle duality of light and matter (see Bohr, 1978a, b). According to Bohr, the wave–particle duality is so central a phenomenon that it should form a natural basis for any interpretation of the quantum theory. Starting from this duality Bohr developed his notion of complementarity which was to "denote the relation of mutual exclusion characteristic to quantum theory with regard to the application of the various classical concepts and ideas" (Bohr, 1978c, p. 19).

Although complementarity has been discussed for more than 50 years, an exact definition of the notion of complementary physical quantities has been found quite recently (Lahti, 1980; see also Lahti, 1979 for a discussion). We assume the following definition, referring the reader to Lahti (1979):

(C) Observables A and B are complementary if for any bounded Borel sets X and Y such that $X \cap \sigma(A) \subsetneq \sigma(A)$ and $Y \cap \sigma(B) \subsetneq \sigma(B)$, the lattice meet $A(X) \wedge B(Y)$ exists in **L** and equals the least element of **L**, 0.

Now we can formulate the complementarity principle:

(CP) There exist at least two nonconstant complementary observables in **O**.

Let us note that the property expressed in (C) is an abstract version of the Hilbert-space result, according to which the Schrödinger couple (Q, P) possesses the property: $Q(X) \wedge P(Y) = 0$ for any bounded X and Y in $\mathbf{B}(\mathbb{R})$ with $Q(X)$ and $P(Y)$ denoting the projections corresponding to X and Y via the spectral decompositions of Q and P, respectively. This property of Q and P emphasizes the complementary nature of position and momentum observables in the Bohr sense (Lahti, 1979).

6. THE QUANTAL NATURE OF THE PRINCIPLES

From the physical point of view it should be clear that each of the above-discussed three principles should in some way reflect the very root of the quantum theory: the existence of the universal quantum of action h. This means that each of the principles should lead to a quantal theory, i.e., to a theory with a typical quantum property. In this chapter we shall shortly check that our formulations of the principles really lead to nonclassical theories. For details we refer the reader to Lahti (1979).

In Section 3 we gave two formulations for the superposition principle: the one referring to the state system **S**, the other referring to the logic **L**. It is immediately clear that both of these formulations exclude the classical mechanical description of any physical system. On the one hand, (SP) implies a nonclassical state system **S** simply because in the classical case no pure state can be a superposition of other pure states distinct from it. On the other hand, (SP′) implies a non-Boolean structure for **L**. Really, assuming that **L** is Boolean and that a, a_1, and a_2 are atoms of **L** satisfying the superposition principle, one could conclude that $a = a\wedge(a_1\vee a_2) = (a\wedge a_1)\vee(a\wedge a_2) = 0$, which is a contradiction.

The unitalness of the state system **S** implies, as one can easily show, that the observables satisfying the uncertainty principle are unbounded and noncompatible. This, again, means that the theory which results after restricting the Mackey description with (UP) is nonclassical. It is also of interest to note that if the unitalness of the state system **S** is not assumed one can have a Mackey description (**L**, **S**) with a Boolean **L** and with a full and convex set **S** of states on **L** such that (UP) is satisfied in (**L**, **S**)[2]. In this case the quantal feature of the description is incorporated in the state system **S** only (**S** cannot contain any unit measures on **L**, i.e., classical pure states), and one can interpret (UP) as deforming the classical theory to a quantal theory with modifying the classical notion of state.

Our formulation of the complementarity principle is also of a quantal nature as it leads to the break of the distributive law in **L**. Really, one can show that (CP) implies the following important property for **L**: there exist in **L** at least two propositions, say a and b, such that their meet $a\wedge b$ exists and equals the least element of **L**, but they are not orthogonal. Moreover, we note that nonconstant complementary observables are always noncompatible.

We conclude that our formulations of the superposition principle, the uncertainty principle, and the complementarity principle are of quantal nature, each of them being enough to exclude the classical mechanical description of the given physical system.

Finally, we note that our formulations of the three principles are logically independent (see Lahti, 1979). This means that they really have the same "logical status" on the foundations of quantum theory. Thus it appears to be only a matter of taste to elevate one of them to the first principle of the theory and to put the other two on a subordinate level. However, this tendency is quite strong in the relevant literature. A typical example of this attitude in modern texts is the book of Varadarajan, where one can read the confession that "the principle of superposition of states is the fundamental concept on which the quantum theory of atomic systems is to be erected" (Varadarajan, 1968, p. xi).

7. THE THREE PRINCIPLES AND THE HILBERTIAN MODEL OF QUANTUM THEORY

Now the following question can be posed: Are the three fundamental principles of quantum theory strong enough to restrict the general Mackey

[2]Such an example was discovered by S. Bugajski and it is discussed in Lahti (1979).

description to the standard Hilbertian quantum mechanics? The answer is in the negative, because of the following counterexample.

Let $(\mathbf{L}^{(1)}, \mathbf{S}^{(1)})$ and $(\mathbf{L}^{(2)}, \mathbf{S}^{(2)})$ be two Hilbertian Mackey descriptions, i.e., $\mathbf{L}^{(i)}$ is the lattice of all orthogonal projections on a separable complex Hilbert space $\mathbf{H}^{(i)}$, whereas $\mathbf{S}^{(i)}$ is the set of all probability measures on $\mathbf{L}^{(i)}, i=1,2$. Let $\mathbf{L}$ be the horizontal sum (Holland, 1970) of $\mathbf{L}^{(1)}$ and $\mathbf{L}^{(2)}, \mathbf{L}=\mathbf{L}^{(1)} \textcircled{H} \mathbf{L}^{(2)}$. One can visualize $\mathbf{L}$ as $\mathbf{L}^{(1)}$ and $\mathbf{L}^{(2)}$ "pasted together" in two "points": $e^{(1)} \leftrightarrow e^{(2)}, 0^{(1)} \leftrightarrow 0^{(2)}$ (where $e^{(i)}$ is the greatest, and $0^{(i)}$ the least element of $\mathbf{L}^{(i)}, i=1,2$). It is evident that $\mathbf{L}$ is an irreducible orthomodular complete lattice (for a general proof see Greechie, 1968), and that probability measures on $\mathbf{L}$ are in a natural correspondence with pairs $(\alpha^{(1)}, \alpha^{(2)}), \alpha^{(i)} \in \mathbf{S}^{(i)}, i=1,2$. The resulting structure $(\mathbf{L}, \mathbf{S})$ with $\mathbf{S}=\mathbf{S}^{(1)} \times \mathbf{S}^{(2)}$ (the Cartesian product) provides an example of both non-Hilbertian and nonclassical Mackey system. The nonclassical character of $(\mathbf{L}, \mathbf{S})$ needs no comments. Its non-Hilbertian character means that $\mathbf{L}$ cannot be represented as the lattice of all orthogonal projections on a Hilbert space.

Observe that the horizontal sum $\mathbf{L}=\mathbf{L}^{(1)} \textcircled{H} \mathbf{L}^{(2)}$ can be canonically embedded into the Hilbertian structure $(\mathbf{L}^{(1)} \otimes \mathbf{L}^{(2)}, \mathbf{S}^{(1)} \otimes \mathbf{S}^{(2)})$ corresponding to the tensor product $\mathbf{H}^{(1)} \otimes \mathbf{H}^{(2)}$: $a^{(1)} \mapsto a^{(1)} \otimes e^{(2)}, a^{(2)} \mapsto e^{(1)} \otimes a^{(2)}, (\alpha^{(1)}, \alpha^{(2)}) \mapsto \alpha^{(1)} \otimes \alpha^{(2)}$. This embedding, however, is not a lattice homomorphism. Thus, e.g., $a^{(1)} \otimes e^{(2)}, e^{(1)} \otimes a^{(2)}$ are compatible in $\mathbf{L}^{(1)} \otimes \mathbf{L}^{(2)}$, and their lattice meet equals $a^{(1)} \otimes a^{(2)}$, whereas $a^{(1)}$ and $a^{(2)}$ for $a^{(1)} \neq e^{(1)}, 0^{(1)}, a^{(2)} \neq e^{(2)}, 0^{(2)}$ are not compatible in $\mathbf{L}^{(1)} \textcircled{H} \mathbf{L}^{(2)}$, and their lattice meet equals 0. On the other hand, our $\mathbf{L}^{(1)} \textcircled{H} \mathbf{L}^{(2)}$ has nothing in common with the direct sum of $\mathbf{H}^{(1)}$ and $\mathbf{H}^{(2)}$, so its specific structure cannot be interpreted as a result of a superselection rule.

Let us consider now the three principles in the case of our $(\mathbf{L}, \mathbf{S})$. It is easy to see that any pure state of $\mathbf{S}$ restricted to $\mathbf{L}^{(i)}$ defines a pure state, say $\alpha^{(i)}$, on $\mathbf{L}^{(i)}$ $(i=1,2)$, and vice versa: any pair $(\alpha^{(1)}, \alpha^{(2)})$ of pure states with $\alpha^{(1)} \in \mathbf{P}^{(1)}$ and $\alpha^{(2)} \in \mathbf{P}^{(2)}$ defines a pure state on $\mathbf{L}$. Now (SP) demands that for any two pure states on $\mathbf{L}$, say $(\alpha^{(1)}, \alpha^{(2)}), (\beta^{(1)}, \beta^{(2)})$, there exists at least one pure state $(\gamma^{(1)}, \gamma^{(2)})$ on $\mathbf{L}$ such that $(\alpha^{(1)}, \alpha^{(2)})(a)=(\beta^{(1)}, \beta^{(2)})(a)=0$ implies $(\gamma^{(1)}, \gamma^{(2)})(a)=0$ for any a in $\mathbf{L}$. It is evident that this implication holds for any pair $(\gamma^{(1)}, \gamma^{(2)})$ such that the Ψ-function corresponding to $\gamma^{(i)}$ is a normalized superposition of those corresponding to $\alpha^{(i)}$ and $\beta^{(i)} (i=1,2)$, hence the set of such $(\gamma^{(1)}, \gamma^{(2)})$ is not empty for any pair of pure states on $\mathbf{L}$.

The mentioned correspondence between atoms and pure states is broken in our $(\mathbf{L}, \mathbf{S})$. Nevertheless (SP′) holds, too. Indeed, the set of atoms of $\mathbf{L}$ is a set sum of the sets of atoms of $\mathbf{L}^{(1)}$ and $\mathbf{L}^{(2)}$. Hence we have only to check if for any two atoms $a^{(1)}$ in $\mathbf{L}^{(1)}$ and $a^{(2)}$ in $\mathbf{L}^{(2)}$ there is a third atom, say a, in $\mathbf{L}$ such that $a \leqslant a^{(1)} \vee a^{(2)}$. It does exist, of course, because $a^{(1)} \vee a^{(2)} = e$ in this case.

The two other principles are satisfied both in $(\mathbf{L}^{(1)},\mathbf{S}^{(1)})$ and in $(\mathbf{L}^{(2)},\mathbf{S}^{(2)})$ separately, so they hold also for $(\mathbf{L},\mathbf{S})$. Moreover, any two observables $A^{(1)}\in\mathbf{O}^{(1)},A^{(2)}\in\mathbf{O}^{(2)}$ are complementary in the sense of Section 5.

Thus we see that the Mackey description supplemented by the abstract form of the superposition principle, the uncertainty principle, and the complementarity principle is still far from being the standard quantum mechanics. In the presence of this example two alternative definitions of the proper quantum mechanical description can be taken into account:

(QM1) The proper quantum description is the theory resulting from the Mackey description after supplementing it by the three mentioned principles.

(QM2) The proper quantum description is provided by the standard Hilbert-space quantum mechanics.

We have demonstrated that the two definitions are not equivalent, hence the Hilbert space is too specific to reflect general ideas of the quantum theory. Accepting (QM1) we have, however, to develop a new, more general representation theory for the resulting form of the Mackey description. On the other hand, if we accept (QM2), we must find at least one additional fundamental principle to obtain the Hilbertian model of general quantum theory. The latter problem will be considered in the next section.

8. THE PROJECTION POSTULATE AND THE STANDARD QUANTUM THEORY

What we want to consider now is essentially the problem of deriving the famous seventh axiom of Mackey from basic principles. It was for many years the main problem of research on the foundations of quantum theories. An analysis of numerous representation theorems for quantum logic, intended to provide a solution to this problem, shows that their common and important constituent is a more or less disguised form of the von Neumann projection postulate (see Bugajska and Bugajski, 1973b). So we guess that this postulate serves as a missing fundamental principle which makes it possible to restrict the Mackey description to the standard Hilbertian quantum theory. It is not quite obvious, however, what form the projection postulate should take in the abstract context of the Mackey description. Hence our first goal is to formulate this principle outside the Hilbert-space operators.

The projection postulate is not related to numerical results of measurements, but rather to the changes of states caused by measurements of some very special type (ideal measurements of the first kind). So we must introduce new elements into the Mackey scheme: the set of operations, or at least its subset called the set of filters. What we need now is a definition of a filter.

The most fundamental property of filters is that they transform pure states into pure states with corresponding decrease of "intensity of the incoming beam".[3] To describe this we define a set $\tilde{\mathbf{P}}$ as the product: $\tilde{\mathbf{P}}=[0,1]\times\mathbf{P}=\{\lambda\alpha|\lambda\in[0,1],\alpha\in\mathbf{P}\}$ with all 0α identified and denoted by ω, and with $\{1\alpha|\alpha\in\mathbf{P}\}$ naturally identical to $\mathbf{P}$. We introduce the "intensity functional" $e:\tilde{\mathbf{P}}\to[0,1]$ by $e(\lambda\alpha)=\lambda, \alpha\in\mathbf{P}$. If $\phi:\tilde{\mathbf{P}}\to\tilde{\mathbf{P}}$ is a filter, and $\alpha\in\mathbf{P}$, then $e(\phi\alpha)$ is the "transmission coefficient" for the selection process described by ϕ.

The set $\mathbf{L}_f$ of filters will be defined by the following properties:

(F1) $e(\phi\alpha)\leqslant e(\alpha), \quad \forall\alpha\in\tilde{\mathbf{P}}, \quad \forall\phi\in\mathbf{L}_f \quad \phi:\tilde{\mathbf{P}}\to\tilde{\mathbf{P}}, \quad \forall\phi\in\mathbf{L}_f$

(F2) $\phi(\lambda\alpha)=\lambda\phi(\alpha), \quad \forall\alpha\in\tilde{\mathbf{P}}, \quad \forall\lambda\in[0,1], \quad \forall\phi\in\mathbf{L}_f$

(F3) $e(\phi\alpha)=e(\alpha)\to\phi\alpha=\alpha, \quad \forall\alpha\in\tilde{\mathbf{P}}, \forall\phi\in\mathbf{L}_f$

(F4) $\phi\alpha'=\alpha', \quad \forall\phi\in\mathbf{L}_f, \alpha\in\mathbf{P}$, where α' is the "normalized" state obtained after selecting the beam corresponding to α by means of the filtering device $\phi, \alpha'=(e(\phi\alpha))^{-1}\phi\alpha$.

(F5) for every $\alpha\in\mathbf{P}$ there exists one and only one filter, denoted by ϕ_α, such that $\phi_\alpha\beta=\beta\to\beta=\alpha, \forall\beta\in\mathbf{P}$

(F6) $e(\phi\alpha)=e(\phi_{\alpha'}\alpha), \forall\phi\in\mathbf{L}_f, \forall\alpha\in\mathbf{P}$, where α' is defined in (F4).

The first property has been commented on briefly above. The second one means, intuitively speaking, that the action of ϕ is independent of the intensity of the incoming beam. It corresponds to the linearity of operations, usually assumed in the operational quantum mechanics (Davies and Lewis, 1970; Mielnik, 1969).

Our (F3) is frequently assumed (see, e.g., Mielnik, 1969; Pool, 1968) and is usually interpreted as the principle of minimal disturbance considered by Lüders (1951) (see also Herbut, 1969). It is evident that (F3) does not exhaust the whole content of this principle, e.g., our properties (F4) and (F6) describes also some aspects of the minimal disturbance principle, as will be discussed below.

It is important to note that (F4) [together with (F1), (F2), and (F3)] implies the idempotency of the filters: $\phi^2=\phi$ for any $\phi\in\mathbf{L}_f$. It means that

[3]For convenience we have adopted here the traditional terminology of the operational quantum mechanics. However, we emphasize that this terminology does not imply a commitment to some specific interpretation of probability in quantum theory.

the projected state $\phi\alpha$ is an eigenstate of ϕ. Hence filters are measurements of the first kind, which is in full agreement with von Neumann (1955).

Our property (F5) expresses the common belief that any pure state can be produced by a selection process. A similar assumption is common in many discussions concerning the foundations of quantum mechanics; see, e.g., Gunson (1967), Pool (1968), Jauch and Piron (1969), and Ochs (1972). Observe that the idempotency of filters leads to the following form of $\phi_\alpha : \phi_\alpha \beta = e(\phi_\alpha \beta)\alpha$ for any $\alpha, \beta \in \mathbf{P}$.

The meaning we have attached to the number $e(\phi\alpha)$ suggests that $e(\phi_\beta \alpha)$ should be interpreted as a transition ratio from the pure state α to the pure state β. Hence the set $\mathbf{P}$ of pure states obtains an additional structure similar to the one of Mielnik's transition probability space (Belinfante, 1976; see also Bugajska and Bugajski, 1973a). The "transition probability" $p(\alpha, \beta) = e(\phi_\beta \alpha)$ can be considered as a numerical characteristics of a "distance" between, or a "similarity" of pure states α and β: the greater $p(\alpha, \beta)$ the "closer" α and β.

Coming back to the principle of minimal disturbance it is obvious now that it contains a requirement like: $\phi \in \mathbf{L}_f$ maps α onto the closest to α eigenstate of ϕ, i.e., that $p(\alpha, \alpha')$, where $\alpha' = e(\phi\alpha)^{-1}\phi\alpha$, equals the supremum of the set $\{p(\alpha, \beta) | \beta \in \mathbf{P}, \phi\beta = \beta\}$. On the other hand it is reasonable to assume that this supremum equals to $e(\phi\alpha)$ (although the properties of the filters we have assumed are too general to make possible a proof of this). This can serve as a justification of our (F6).

It seems that above properties constitute a minimal requirement to select filters among all operations on states. Other natural properties which are usually imposed on $\mathbf{L}_f$, as, e.g., the existence of a mapping $\phi \mapsto \phi^\perp$ ("orthocomplementation", or "negation") with some natural properties (Davies and Lewis, 1970; Pool, 1968) are justified a posteriori after introducing a more or less explicit form of the projection postulate.

Let us observe now that any filter $\phi \in \mathbf{L}_f$ defines a function $\mathbf{P} \to [0,1], \alpha \mapsto e(\phi\alpha)$, as do the elements of $\mathbf{L}$. These functions ("decision effects" of Ludwig) can be identified on physical grounds with elements of $\mathbf{L}$. It is plausible to assume that any $e(\phi\alpha)$ is represented in $\mathbf{L}$. Conversely, observe that we could associate to any $a \in \mathbf{L}$ a mapping $\tilde{\Omega}_a$ of $\mathbf{P}$ into $a^1 = \{\alpha \in \mathbf{P} | \alpha(a) = 1\}$ by means of the Sasaki projection (compare Cassinelli and Beltrametti, 1975). The corresponding filter Ω_a is defined by $\Omega_a \alpha = \alpha(a)\tilde{\Omega}_a \alpha$. This all can be done if $\mathbf{L}$ is equipped with much stronger properties that in the case of the Mackey description. These stronger additional properties are sometimes identified as an abstract version of the projection postulate (Bugajska and Bugajski, 1973b). It seems, however, that the essential point of the von Neumann–Lüders idea about the change of states under ideal measurements of the first kind can be expressed in the following way:

(PP) There is a natural one-to-one correspondence Ω between the elements of **L** and the elements of $\mathbf{L}_f$ with the property:

$$\alpha(a) = e(\Omega(a)\alpha) \text{ for every } a \in \mathbf{L} \text{ and } \alpha \in \mathbf{P}.$$

This way of understanding the projection postulate is rather new. Similar ideas can be found in Gunson (1967), Pool (1968), Jauch and Piron (1969), Ochs (1972), and Beltrametti and Cassinelli (1977).

The correspondence between questions and filters, assumed in (PP), has a strong influence on the structures of both **L** and $\mathbf{L}_f$. For our purpose the most interesting are new features of the Mackey logic **L** induced by (PP).

Thus (F5) transformed into $(\mathbf{L}, \mathbf{S})$ assures the existence of atoms in **L**. Indeed, if $a \in \mathbf{L}$ and $\alpha \in \mathbf{P}$, then $a \leqslant \Omega^{-1}(\phi_\alpha)$ implies that any eigenstate of $\Omega(a)$ is also an eigenstate of ϕ_α, which means that $\Omega(a) = \phi_\alpha$ by (F5). Hence $\Omega^{-1}(\phi_\alpha)$ is an atom, moreover (F5) defines a natural one-to-one correspondence between atoms of **L** and pure states, so **L** appears to be atomic.

The principle of minimal disturbance in the form of (F6) induces on L the Varadarajan (1968) property. Let a be an atom of **L** and b an element of **L** such that a is not orthogonal to b. By (PP) and (F5) there is a pure state, say α, corresponding to a, and there are filtering operations $\Omega(b)$ and $\Omega(b^\perp)$ corresponding to b and $b^\perp$, respectively. The action of $\Omega(b)$ and $\Omega(b^\perp)$ on α produces two pure states $\beta_1 = (e(\Omega(b)\alpha))^{-1}\Omega(b)\alpha$ and $\beta_2 = (e(\Omega(b^\perp)\alpha))^{-1}\Omega(b^\perp)\alpha$. Obviously $\beta_1(b) = \beta_2(b^\perp) = 1$, and the corresponding atoms of **L**, which will be denoted b_1 and b_2, are contained in b and $b^\perp$ respectively: $b_1 \leqslant b, b_2 \leqslant b^\perp$. The property (F6) says now that the atom a is less than, or equal to, $b_1 \vee b_2$. This is the Varadarajan version of the covering law (see Bugajska and Bugajski, 1973b).

Now the only property of **L** which we need to apply the Piron representation theorem is the complete lattice property. Note that we are very close to it, as for separable **L** it can be demonstrated (see Bugajska and Bugajski, 1973c) that our **L** is a complete lattice. It is not clear if the assumption of separability for **L** has a deeper physical justification. It seems that this property, which is so fundamental for the von Neumann version of quantum mechanics, deserves closer attention, and its eventual physical roots are to be found.

If we do not like to assume the separability of **L**, we can construct a natural embedding of **L** into a complete atomic orthomodular lattice $\tilde{\mathbf{L}}$ (Bugajska and Bugajski, 1973c), which preserves all essential structural features of **L** and does not introduce new atoms. If we extend the projection postulate on this new $\tilde{\mathbf{L}}$, we obtain exactly a lattice satisfying all requirements of the Piron theorem, because the Varadarajan property is equivalent to the covering law if only **L** is a lattice.

Thus we conclude that the projection postulate is almost all we need to obtain the Hilbert-space model of quantum theory. Let us observe that (PP) does not exclude the classical case (although it becomes trivial there), which corresponds to the Hilbert-space description with extremely large set of superselection rules. To exclude this extreme case we must assume that L is not Boolean, or better to postulate some of the previously discussed principles such as, for example, the complementarity principle.

9. CONCLUDING REMARKS

In this paper we have shown that the Dirac–Heisenberg–Bohr quantum theory (i.e., the theory resulting from the Mackey description after supplementing it with the superposition principle, the uncertainty principle, and the complementarity principle) and the von Neumann quantum theory (i.e., the standard Hilbert-space quantum theory) are not equivalent, the former being more general. In order to reach the von Neumann quantum theory from the Mackey description we introduced a version (PP) of the projection postulate, and demonstrated that the projection postulate is, apart from a trivial case, essentially enough to do the job. This result, however, puts the three fundamental principles of the theory in a very odd light: they have merely the modest duty of excluding the trivial case of the Hilbert-space description with an extremely large set of superselection rules.

Though admittedly not quite satisfactory, with the above results we are allured into thinking that in the early years two different quantum theories were developed, namely: the Dirac–Heisenberg–Bohr quantum theory, which is based on the superposition principle, the uncertainty principle, and the complementarity principle and the von Neumann quantum theory, which is based on the projection postulate. The former was developed by the great fathers of the theory, whereas the latter was developed by von Neumann (1955) in his classic treatise. Though, of course, the von Neumann quantum theory is the one that physicists use in their practical work (calculations), it seems that the Dirac–Heisenberg–Bohr quantum theory is the theory that physicists, mostly in the early years of quantum theory, have in mind in discussing philosophical problems (like interpretation) of the theory (see, e.g., Jammer, 1966, 1974).

Finally, we recall that our analysis of the derivability of the Hilbertian quantum theory from the Mackey description supplemented with (PP) revealed the importance of the question of the separability of the Mackey logic L. Thus, probably, "the separability principle" should be added to the collection of the fundamental principles of the theory. However, the physical roots of this principle still wait to be found.

REFERENCES

Belinfante, J. G. F. (1976). *Journal of Mathematical Physics*, **17**, 285–291.

Beltrametti, E. G., and Cassinelli, G. (1977). *Journal of Philosophy and Logic*, **6**, 369–379.

Bohr, N. (1978a). "Atomic Theory and Mechanics," reprinted in *Atomic Theory and the Description of Nature*, AMS Press, New York.

Bohr, N. (1978b). "The Quantum Postulate and the Recent Development of Atomic Theory," reprinted in *Atomic Theory and the Description of Nature*. AMS Press, New York.

Bohr, N. (1978c). "Introductory Survey," reprinted in *Atomic Theory and the Description of Nature*. AMS Press, New York.

Bugajska, K., and Bugajski, S. (1973a). *Reports on Mathematical Physics*, **4**, 1–20.

Bugajska, K., and Bugajski, S. (1973b). *Bulletin de l'Academie Polonaise des Sciences, Serie des Sciences Mathematiques, Astronomiques et Physiques*, **21**, 873–877.

Bugajska, K., and Bugajski, S. (1973c). *Annales de l'Institut Henri Poincaré*, **19**, 333–340.

Cantoni, V. (1976). *Communications in Mathematical Physics*, **50**, 241–244.

Cassinelli, G., and Beltrametti, E. G. (1975). *Communications in Mathematical Physics*, **40**, 7–13.

Catlin, D. E. (1968). *International Journal of Theoretical Physics*, **1**, 285–297.

Davies, E. B., and Lewis, J. T. (1970). *Communications in Mathematical Physics*, **17**, 239–260.

Deliyannis, P. C. (1976). *Journal of Mathematical Physics*, **17**, 248–254.

Dirac, P. A. M. (1958). *The Principle of Quantum Mechanics*. Oxford University Press, Amen House, London E.C.4.

Fischer, H. R., and Rüttimann, G. T. (1978). in *Mathematical Foundations of Quantum Theory*, A. R. Marlow, ed., pp. 153–176. Academic Press, New York.

Greechie, R. J. (1968). *Journal of Combination Theory*, **4**, 210–218.

Gudder, S. P. (1970). *Journal of Mathematical Physics*, **11**, 1037–1040.

Gunson, J. (1967). *Communications in Mathematical Physics*, **6**, 262–285.

Haag, R., and Bannier, U. (1978). *Communications in Mathematical Physics*, **60**, 1–6.

Heisenberg, W. (1927). *Zeitschrift für Physik*, **43** 172–198.

Heisenberg, W. (1949). *The Physical Principles of the Quantum Theory*. Dover, New York.

Herbut, F. (1969). *Annals of Physics*, **55**, 271–300.

Holland, S. S., Jr. (1970). in *Trends in Lattice Theory*, J. C. Abbot, ed. Van Nostrand–Reinhold Company, New York.

Jammer, M. (1966). *The Conceptual Foundations of Quantum Mechanics*. McGraw-Hill, New York.

Jammer, M. (1974). *The Philosophy of Quantum Mechanics*. John Wiley & Sons, New York.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*. Addison-Wesley, Reading, Massachusetts.

Jauch, J. M., and Piron, C. (1969). *Helvetica Physica Acta*, **42**, 842–848.

Lahti, P. J. (1979). "Uncertainty and Complementarity in Axiomatic Quantum Mechanics," Doctoral dissertation, University of Turku. Will also appear in this journal.

Lahti, P. J. (1980). *Reports on Mathematical Physics*, in print.

Lüders, G. (1951). *Annalen der Physik*, **8**, 322–328.

Mackey, G. W. (1963). *The Mathematical Foundations of Quantum Mechanics*. Benjamin, New York.

Maczynski, M. J. (1967). *Bulletin de l'Academie Polonaisedes Sciences, Series des Sciences, Mathematiques, Astronomiques et Physiques*, **15**, 583–587.

Mielnik, B. (1969). *Communications in Mathematical Physics*, **15**, 1–46.

Mielnik, B. (1974). *Communications in Mathematical Physics*, **37**, 221–256.

Ochs, W. (1972). *Communications in Mathematical Physics*, **25**, 245–252.

Pool, J. C. T. (1968). *Communications in Mathematical Physics*, **9**, 118–141, 212–228.

Pulmannova, S. (1976). *Communications in Mathematical Physics*, **49**, 47–51.

Robertson, A. P., and Robertson, W. J. (1966). *Topological Vector Spaces*, Cambridge University Press, Cambridge.

Varadarajan, V. S. (1968). *Geometry of Quantum Theory I*, Van Nostrand, New York.

von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, Princeton, New Jersey.

Zabey, P. C. (1975). *Foundations of Physics*, **5**, 323–342.